

Gaudin magnet with impurity and its generalized Knizhnik-Zamolodchikov equation

A. Lima-Santos and Wagner Utiel

*Universidade Federal de São Carlos, Departamento de Física
Caixa Postal 676, CEP 13569-905 São Carlos, Brazil*

Abstract

This work is concerned with the formulation of the boundary quantum inverse scattering method for the xxz Gaudin magnet coupled to boundary impurities with arbitrary exchange constants. The Gaudin magnet is diagonalized by taking a quasi-classical limit of the inhomogeneous lattice. Using the method proposed by Babujian, the integral representation for the solution of the Knizhnik-Zamolodchikov equation is explicitly constructed and its rational limit discussed.

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1 Introduction

Integrable quantum field theories with boundaries have been subject of intense study during the past decades. The great interest in such theories stems from the large number of potential applications in different areas in physics, including open strings, boundary conformal field theory, dissipative quantum phenomena and impurity problems.

The Gaudin magnet has its origins in [1] as a quantum integrable model describing N spin- $\frac{1}{2}$ particles with long-range interactions. The Gaudin type models have direct applications in condensed matter physics. They also have been used as a testing ground for ideas such as the functional Bethe ansatz (BA) and general procedure of separation of variables [2, 3, 4].

The model proposed by Gaudin was later generalized by several authors [5, 6, 7]. The spin-s XXY Gaudin model was solved in [8] by means of the off-shell algebraic BA.

The XYZ Gaudin model was constructed and solved in [3] and [9] by means of the algebraic BA method. The boundary XXY spin- $\frac{1}{2}$ Gaudin magnet was investigated by Hikami [10] and the Gaudin models based on the face-type elliptic quantum groups and boundary elliptic quantum group, as well as, the boundary XYZ Gaudin models were studied in [11] by means of the boundary algebraic BA method. In [12] the XXZ Gaudin model was solved with generic integrable boundaries specified by generic non-diagonal K -matrices.

The Knizhnik-Zamolodchikov (KZ) equations were first proposed as a set of differential equations satisfied by correlation functions of the Wess-Zumino-Witten models [13]. The relations between the Gaudin magnets and the KZ equations has been studied in many papers [8, 14, 15, 16, 17, 18]. In [10], Hikami gave an integral representation for the solutions of the KZ equations by using the results of the boundary Gaudin model.

In addition, the quantum impurity problem, which has been extensively investigated with renormalizing group techniques [19] and conformal field theory [20], is also very interesting in itself. Andrei and Johannesson [21] first considered an impurity spin-s embedded in an integrable spin- $\frac{1}{2}$ XXX chain with periodic boundary conditions. Subsequently, Schlottmann et al [22] generalized it to the arbitrary spin chain. The standard approach to dealing with the impurity integrable problem is also the algebraic BA method. The Hamiltonian of the impurity integrable spin chain can be constructed from the inhomogeneous transfer matrix. The key point is to find some inhomogeneous vertex matrices, which satisfy the same Yang-Baxter relation of the homogeneous matrices, corresponding to impurity spins.

We note that Sklyanin [7], Mezincescu and Nepomechie [24] have used a constant number K matrix to construct their model, where the K matrix induces the boundary fields and boundary bound states [23, 25]. In [26, 27], Wang and coworkers first introduced the operator K matrix to study the Kondo problem in one-dimensional strongly correlated electron systems. In a previous paper [28], the problem

of an open spin- $\frac{1}{2}$ Heisenberg chain coupled to two spin- s impurities sited at the ends has been studied. Following this idea, Shu Chen *et al* [29] have considered de XXZ chain coupled to impurity spins with different coupling constants on the boundary.

In this paper, we continue to study the XXZ chain in order to establish a link between the Gaudin model and the impurity problem. In the first part we construct the eigenstate of the Gaudin magnet with impurity by taking a quasi-classical limit of the transfer matrix for the inhomogeneous open spin chain. The Hamiltonian is given as a solution of the classical Yang-Baxter equation. In the second part, the KZ equation is studied in the context of the impurity problem; the integral solution of the KZ equation is obtained in terms of the Bethe eigenstate of the Gaudin magnet with impurity.

This paper is organized as follows. In section 2 the boundary algebraic Bethe ansatz is reviewed. The transfer matrix for $su(2)$ spin- $\frac{1}{2}$ XXZ chain is construct in terms of the R -matrix and the operator K matrix. In section 3 we find the Hamiltonian of the Gaudin magnet with impurity by taking a quasi-classical limit of the double row transfer matrix with operator K matrices and subsequently its spectrum is obtained. In section 4 the off-shell Bethe ansatz of Babujian is used to find the explicit integral solution of the KZ equation. The section 5 is reserved to a summary and discussion.

2 The quantum inverse scattering method

It is well know that in an integrable problem the quantum R -matrix satisfies the Yang-Baxter equation (YBE):

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \quad (2.1)$$

As usual, R_{ij} means the matrix on $V^{(1)} \otimes V^{(2)} \otimes V^{(3)}$ acting on the i th and j th spaces and as an identity on the other spaces. The variables u and v are called the spectral parameters. As a solution of (2.1), we use the R -matrix for the six-vertex model defined as

$$R(u, \eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u, \eta) & c(u, \eta) & 0 \\ 0 & c(u, \eta) & b(u, \eta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.2)$$

where

$$b(u, \eta) = \frac{\sinh u}{\sinh(u + \eta)}, \quad c(u, \eta) = \frac{\sinh \eta}{\sinh(u + \eta)}. \quad (2.3)$$

Note that in addition to the spectral parameter u , we have a deformation parameter η which parametrizes the anisotropy. Moreover, this solution has the following properties:

$$\begin{aligned} \text{regularity} &: R(u = 0, \eta) = P \\ \text{quasi-classical condition} &: R(u, \eta = 0) = 1 \\ \text{unitarity} &: R(u, \eta)R(-u, \eta) = 1 \end{aligned} \quad (2.4)$$

where P is the permutation operator: $P|\alpha\rangle\otimes|\beta\rangle=|\beta\rangle\otimes|\alpha\rangle$ and t_i denotes the transposition of the i th space.

Let us define the monodromy matrix $T(u)$ for the inhomogeneous N -sites spin chain, introducing the inhomogeneity in the lattice through the parameter $z_N \in C$, by

$$T_0(u|z) = R_{0N}(u - z_N) \cdots R_{02}(u - z_2)R_{01}(u - z_1) = \begin{pmatrix} A(u|z) & B(u|z) \\ C(u|z) & D(u|z) \end{pmatrix} \quad (2.5)$$

Here the operator matrix elements act on the full Hilbert space $V^{\otimes N}$. Due to the additive property of the spectral parameter, the YBE also holds for the inhomogeneous lattice and we have

$$R_{12}(u - v) [T(u|z) \otimes T(v|z)] = [T(v|z) \otimes T(u|z)] R_{12}(u - v) \quad (2.6)$$

Equation (2.6) gives the fundamental algebraic structure for the QISM and gives us the commutation relations between the operators $A(u|z), B(u|z), C(u|z)$ and $D(u|z)$

$$\begin{aligned} [B(u|z), B(v|z)] &= 0 \\ A(u|z)B(u|z) &= \frac{1}{b(v-u)}B(v|z)A(u|z) - \frac{c(v-u)}{b(v-u)}B(u|z)A(v|z) \\ D(u|z)B(v|z) &= \frac{1}{b(u-v)}B(v|z)D(u|z) - \frac{c(u-v)}{b(u-v)}B(u|z)D(v|z) \\ [B(u|z), C(v|z)] &= \frac{c(u-v)}{b(u-v)}(D(v|z)A(u|z) - D(u|z)A(v|z)) \end{aligned} \quad (2.7)$$

To construct the eigenstate of our system, we can use the reference state $|0\rangle$

$$|0\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_1 \otimes \cdots \otimes \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_N \quad (2.8)$$

It is easy to check that this state is the eigenstate of the operators $A(u|z)$ and $D(u|z)$, and annihilated by $C(u|z)$:

$$A(u|z)|0\rangle = |0\rangle, \quad D(u|z)|0\rangle = \prod_{k=1}^N \frac{\sinh(u - z_k)}{\sinh(u - z_k + \eta)}|0\rangle, \quad C(u|z)|0\rangle = 0 \quad (2.9)$$

We note that the Bethe state, a sum of spin waves [30], can be generated by operators $B(u|z)$ acting on the reference state $|0\rangle$.

In order to construct an integrable open chain with boundary impurities, it is necessary to introduce the reflection matrices $K^-(u)$ and $K^+(u)$ which satisfy the following reflecting equations [7]:

$$R_{12}(u - v)K_1^-(u)R_{12}(u + v)K_2^-(v) = K_2^-(v)R_{12}(u + v)K_1^-(u)R_{12}(u - v) \quad (2.10)$$

$$R_{12}(-u + v)K_1^{+t_1}(u)R_{12}(-u - v - 2\eta)K_2^{+t_2}(v) = K_2^{+t_2}(v)R_{12}(-u - v - 2\eta)K_1^{+t_1}(u)R_{12}(-u + v) \quad (2.11)$$

The solution $K_1^\pm = K^\pm \otimes I$ and $K_2^\pm = I \otimes K^\pm$ are the simplest reflection matrices which satisfies the reflecting equations. The inhomogeneous transfer matrix $t(u)$ is defined as

$$t(u|z) = \text{Tr}_0(K_0^+(u)T_0(u|z)K_0^-(u)T_0^{-1}(-u|z)) \quad (2.12)$$

and forms a one-parameter commutative family

$$[t(u|z), t(v|z)] = 0 \quad (2.13)$$

where the monodromy matrix $T(u|z)$ is given by (2.5) and by virtue of the unitarity property of our R -matrix (2.4), it follows the expression for the reflected monodromy matrix

$$T_0^{-1}(-u|z) = R_{01}(u+z_1)R_{02}(u+z_2)\cdots R_{0N}(u+z_N) = \begin{pmatrix} A(u|z) & B(u|z) \\ C(u|z) & D(u|z) \end{pmatrix} \quad (2.14)$$

It was proved in [29] that if τ obeys the Yang relation

$$R_{12}(u-v)\tau_1(u)\tau_2(v) = \tau_2(u)\tau_1(v)R_{12}(u-v) \quad (2.15)$$

then

$$K^-(u) = \tau(u+c)\tau^{-1}(-u+c) \quad (2.16)$$

also obeys the reflecting equation (2.10) and c is a constant.

Therefore we can construct our reflection matrix as

$$K_0^-(u) = R_{0L}(u+c_L)R_{0L}^{-1}(-u+c_L) = R_{0L}(u+c_L)R_{0L}(u-c_L) \quad (2.17)$$

where c_L is a constant decided by the left boundary. This construction give us an operator K -matrix instead of a constant numerical matrix, where it is identified as impurity and is not a pure reflection.

In order to obtain the integrable Hamiltonian in the open chain we define

$$U_a(u|z) = K_0^+(u)\mathcal{M}_0(u|z)K_0^-(u)\mathcal{M}_0^{-1}(-u|z) \quad (2.18)$$

where $K_0^+(u) = I$ and $K_0^-(u)$ is given by (2.17), and $\mathcal{M}_0(u|z)$, $\mathcal{M}_0^{-1}(-u|z)$ are defined as

$$\mathcal{M}_0(u|z) = R_{0R}(u+c_R)T_0(u|z), \quad \mathcal{M}_0^{-1}(u|z) = T_0^{-1}(-u|z)R_{0R}(u-c_R) \quad (2.19)$$

Here c_R is a constant decided by the right boundary. Now one can prove that $U(u)$ satisfies the reflecting equation

$$R_{12}(u-v)U_1(u)R_{12}(u+v)U_2(v) = U_2(v)R_{12}(u+v)U_1(u)R_{12}(u-v). \quad (2.20)$$

In order to derive the corresponding Hamiltonian we first recover the homogeneous case taking $z_i = 0, i = 1,..N$ and the new transfer matrix

$$X(u) = \text{Tr}(K_0^+(u)\mathcal{M}_0(u)K_0^-(u)\mathcal{M}_0^{-1}(-u)) \quad (2.21)$$

The Hamiltonian can be obtained by

$$\begin{aligned} H = \frac{d}{du} \ln X(u)|_{u=0} = & \sum_{j=1}^{N-1} (\sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- + \cosh \eta \sigma_j^z \sigma_{j+1}^z) \\ & + \cosh c_L (\sigma_1^- \sigma_L^+ + \sigma_1^+ \sigma_L^-) + \cosh \eta \sigma_1^z \sigma_L^z \\ & + \cosh c_R (\sigma_N^- \sigma_R^+ + \sigma_N^+ \sigma_R^-) + \cosh \eta \sigma_N^z \sigma_R^z \end{aligned} \quad (2.22)$$

This Hamiltonian describes, beside the bulk, the interaction of the particles with the impurities in the magnetic system. The contribution of the right and left impurities is explicitly in the form of the Hamiltonian.

In the following we will use the algebraic Bethe ansatz [7] to solve the spectrum of $X(u)$ for the inhomogeneous N -site spin chain.

The double row monodromy matrix can be written as

$$U_a(u|z) = \begin{pmatrix} \mathcal{A}(u|z) & \mathcal{B}(u|z) \\ \mathcal{C}(u|z) & \mathcal{D}(u|z) \end{pmatrix} \quad (2.23)$$

Note that the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} act on the Hilbert space $V^{\otimes N}$. The boundary Yang-Baxter relation (2.10) can be rewritten in terms of the operator matrix elements of $U_a(u|z)$. For convenience we define the operator

$$\hat{\mathcal{D}}(u|z) = \sinh(2u + \eta)\mathcal{D}(u|z) - \sinh \eta \mathcal{A}(u|z). \quad (2.24)$$

then we will have the following commutation relations

$$[\mathcal{B}(u|z), \mathcal{B}(v|z)] = 0 \quad (2.25)$$

$$\begin{aligned} \mathcal{A}(u|z)\mathcal{B}(v|z) = & \frac{\sinh(u+v)\sinh(u-v-\eta)}{\sinh(u+v+\eta)\sinh(u-v)}\mathcal{B}(v|z)\mathcal{A}(u|z) \\ & + \frac{\sinh(2v)\sinh \eta}{\sinh(2v+\eta)\sinh(u-v)}\mathcal{B}(u|z)\mathcal{A}(v|z) \\ & - \frac{\sinh \eta}{\sinh(u+v+\eta)\sinh(2v+\eta)}\mathcal{B}(u|z)\hat{\mathcal{D}}(v|z) \end{aligned} \quad (2.26)$$

$$\begin{aligned} \hat{\mathcal{D}}(u|z)\mathcal{B}(v|z) = & \frac{\sinh(u-v-\eta)\sinh(u+v+2\eta)}{\sinh(u-v)\sinh(u+v+\eta)}\mathcal{B}(v|z)\hat{\mathcal{D}}(u|z) \\ & - \frac{\sinh(2u+2\eta)\sinh \eta}{\sinh(2v+\eta)\sinh(u-v)}\mathcal{B}(u|z)\hat{\mathcal{D}}(v|z) \\ & + \frac{\sinh \eta \sinh(2v)\sinh(2u+2\eta)}{\sinh(u+v+\eta)\sinh(2v+\eta)}\mathcal{B}(u|z)\mathcal{A}(v|z) \end{aligned} \quad (2.27)$$

The transfer matrix $X(u)$ can be expressed as

$$\begin{aligned} X(u|z) &= \mathcal{A}(u|z) + \mathcal{D}(u|z) \\ &= \frac{\sinh(2u + \eta) + \sinh \eta}{\sinh(2u + \eta)} \mathcal{A}(u|z) + \frac{1}{\sinh(2u + \eta)} \hat{\mathcal{D}}(u|z) \end{aligned} \quad (2.28)$$

Now we define a reference state by including the two boundary impurities sites

$$|0\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_L \otimes \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_1 \otimes \cdots \otimes \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_N \otimes \left(\begin{array}{c} 1 \\ 0 \end{array} \right)_R \quad (2.29)$$

This reference state is an eigenstate of $\mathcal{A}(u|z)$ and $\hat{\mathcal{D}}(u|z)$ and annihilated by $\mathcal{C}(u|z)$

$$\mathcal{A}(u|z) |0\rangle = |0\rangle, \quad \hat{\mathcal{D}}(u|z) |0\rangle = \hat{d}(u|z) |0\rangle, \quad \mathcal{C}(u|z) |0\rangle = 0 \quad (2.30)$$

where

$$\begin{aligned} \hat{d}(u|z) &= 2 \sinh(u) \cosh(u + \eta) \prod_{a=L,R} \frac{\sinh(u + c_a) \sinh(u - c_a)}{\sinh(u + c_a + \eta) \sinh(u - c_a + \eta)} \\ &\quad \times \left[\prod_{k=1}^N \frac{\sinh(u - z_k) \sinh(u + z_k)}{\sinh(u - z_k + \eta) \sinh(u + z_k + \eta)} \right] \end{aligned} \quad (2.31)$$

The eigenstate of $X(u)$ with M spins down is given by the Bethe state

$$\Psi(\{u\}) = \prod_{a=1}^M \mathcal{B}(u_a|z) |0\rangle \quad (2.32)$$

Using the commutation relations between operators $\mathcal{A}(u|z)$, $\mathcal{B}(u|z)$ and $\hat{\mathcal{D}}(u|z)$, we obtain

$$X(u)\Psi(\{u\}) = \Lambda(u; \{u\}|z)\Psi(\{u\}) + \sum_a \mathcal{F}_a \Psi^a(\{u\}) \quad (2.33)$$

where

$$\begin{aligned} \Lambda(u; \{u\}|z) &= \frac{\sinh(2u + \eta) + \sinh \eta}{\sinh(2u + \eta)} \prod_{a=1}^M \frac{\sinh(u - u_a - \eta) \sinh(u + u_a)}{\sinh(u - u_a) \sinh(u + u_a + \eta)} \\ &\quad + \frac{\hat{d}(u|z)}{\sinh(2u + \eta)} \prod_{a=1}^M \frac{\sinh(u - u_a + \eta) \sinh(u + u_a + 2\eta)}{\sinh(u - u_a) \sinh(u + u_a + \eta)} \end{aligned} \quad (2.34)$$

$$\begin{aligned} \mathcal{F}_a &= \left\{ \left[\prod_{b \neq a}^M \frac{\sinh(u_a - u_b - \eta) \sinh(u_a + u_b)}{\sinh(u_a - u_b) \sinh(u_a + u_b + \eta)} \right] \sinh(2u_a) \cosh(u_a) \right. \\ &\quad \left. - \left[\prod_{b \neq a}^M \frac{\sinh(u_a - u_b + \eta) \sinh(u_a + u_b + 2\eta)}{\sinh(u_a - u_b) \sinh(u_a + u_b + \eta)} \right] \cosh(u_a + \eta) \hat{d}(u_a|z) \right\} \\ &\quad \times \frac{2 \sinh(u + \eta) \sinh \eta}{\sinh(u - u_a) \sinh(2u_a + \eta) \sinh(u + u_a + \eta)} \end{aligned} \quad (2.35)$$

and

$$\Psi^a(\{u\}) = \mathcal{B}(u|z) \prod_{b \neq a}^M \mathcal{B}(u_b|z) |0\rangle \quad (2.36)$$

The relation (2.33) shows that the Bethe state $\Psi(\{u\})$ is an eigenstate of the transfer matrix $X(u)$ under the condition $\mathcal{F}_a = 0$, $a = 1, \dots, M$, i.e.

$$\begin{aligned} \prod_{b \neq a}^N \frac{\sinh(u_a - z_b + \eta)}{\sinh(u_a - z_b)} \frac{\sinh(u_a + z_b + \eta)}{\sinh(u_a + z_b)} &= \frac{\cosh^2(u_a + \eta)}{\cosh^2 u_a} \\ \times \prod_{j=L,R} \frac{\sinh(u_a + c_j) \sinh(u_a - c_j)}{\sinh(u_a + c_j + \eta) \sinh(u_a - c_j + \eta)} \prod_{b \neq a}^M &\frac{\sinh(u_a - u_b + \eta) \sinh(u_a + u_b + 2\eta)}{\sinh(u_a - u_b - \eta) \sinh(u_a + u_b)} \end{aligned} \quad (2.37)$$

This equation with $z_k = 0$ corresponds to the Bethe ansatz equation for XXZ spin chain with impurities derived in [29].

3 The Gaudin magnet

We will show how the Gaudin magnet with impurity can be derived from the identity (2.33). The Gaudin magnet can be obtained by taking the quasi-classical limit $\eta \rightarrow 0$ of the transfer matrix for the inhomogeneous spin chain [1]. This fact indicates that the Hamiltonian is written in terms of the solution of the classical YBE.

Due to the quasi-classical condition, we have the power series expansion around the point $\eta = 0$ for each term in (2.33,2.34).

$$X(u = z_j) = -\eta + \eta^2 H_j + o(\eta^3), \quad \Lambda(u = z_j) = -\eta + \eta^2 E_j + o(\eta^3), \quad (3.1)$$

and from (2.35) we have

$$\mathcal{F}_a = -\eta^2 \frac{2 \sinh z_j \cosh v_a}{\sinh(z_j - v_a) \sinh(z_j + v_a)} f_a + o(\eta^3) \quad (3.2)$$

where we have the Hamiltonian H_j :

$$\begin{aligned} H_j &: = \sum_{k=1}^N \frac{1}{\sinh(z_j + z_k)} \left\{ \sigma_j^- \sigma_k^+ + \sigma_j^+ \sigma_k^- + \frac{1}{2} \cosh(z_j + z_k) (\sigma_j^z \sigma_k^z + 1) \right\} \\ &+ \sum_{k \neq j, k=1}^N \frac{1}{\sinh(z_j - z_k)} \left\{ \sigma_j^- \sigma_k^+ + \sigma_j^+ \sigma_k^- + \frac{1}{2} \cosh(z_j - z_k) (\sigma_j^z \sigma_k^z + 1) \right\} \\ &+ \frac{2 \sinh z_j}{\sinh(z_j - c_L) \sinh(z_j + c_L)} \left\{ \cosh c_L (\sigma_j^- \sigma_L^+ + \sigma_j^+ \sigma_L^-) + \frac{1}{2} \cosh(z_j) (\sigma_j^z \sigma_L^z + 1) \right\} \\ &+ \frac{2 \sinh z_j}{\sinh(z_j - c_R) \sinh(z_j + c_R)} \left\{ \cosh c_R (\sigma_R^- \sigma_j^+ + \sigma_R^+ \sigma_j^-) + \frac{1}{2} \cosh(z_j) (\sigma_R^z \sigma_j^z + 1) \right\}, \end{aligned} \quad (3.3)$$

the energy E_j :

$$\begin{aligned} E_j &= \frac{1}{\sinh(2z_j)} + \sum_{k=L,R} [\coth(z_j - c_k) + \coth(z_j + c_k)] \\ &\quad - \sum_{a=1}^M [\coth(z_j - v_a) + \coth(z_j + v_a)] \end{aligned} \quad (3.4)$$

and the unwanted factor f_a :

$$\begin{aligned} f_a &= 2 \tanh v_a - \sum_{k=L,R} [\coth(v_a - c_k) + \coth(v_a + c_k)] \\ &\quad + 2 \sum_{\substack{b=1 \\ b \neq a}}^M [\coth(v_a - v_b) + \coth(v_a + v_b)] - \sum_{k=1}^N [\coth(v_a - z_k) + \coth(v_a + z_k)] \end{aligned} \quad (3.5)$$

Note that we have used the notation ($u = z_j$) to mean the residue at $u = z_j$. The integrability of these Hamiltonians follow from their commutativity

$$[H_j, H_k] = 0, \quad j = 1, \dots, N \quad (3.6)$$

which is obtained from the commutativity of the transfer matrix.

The Bethe states Ψ (2.32) and Ψ_a (2.36) have the following expansions

$$\Psi(v) = \eta^M \phi + o(\eta^{M+1}), \quad \Psi^a(z_j) = \eta^{M-1} \sigma_j^- \phi^a + o(\eta^M) \quad (3.7)$$

with

$$\phi = \prod_{a=1}^M \left\{ \mathcal{Z}_a + \sum_{k=1}^N \left(\frac{1}{\sinh(v_a - z_k)} + \frac{1}{\sinh(v_a + z_k)} \right) \sigma_k^- \right\} |0\rangle \quad (3.8)$$

and

$$\phi^a = \prod_{\substack{b=1 \\ b \neq a}}^M \left\{ \mathcal{Z}_b + \sum_{k=1}^N \left(\frac{1}{\sinh(v_b - z_k)} + \frac{1}{\sinh(v_b + z_k)} \right) \sigma_k^- \right\} |0\rangle \quad (3.9)$$

where \mathcal{Z}_a are the impurity contributions

$$\mathcal{Z}_a = \left(\frac{1}{\sinh(v_a - c_L)} + \frac{1}{\sinh(v_a + c_L)} \right) \sigma_L^- + \left(\frac{1}{\sinh(v_a - c_R)} + \frac{1}{\sinh(v_a + c_R)} \right) \sigma_R^- \quad (3.10)$$

When we combine the terms proportional to η^{M+1} in (2.33), we obtain the so-called off-shell Bethe ansatz equation

$$H_j \phi = E_j \phi + \sum_{a=1}^M \frac{2 \sinh z_j \cosh v_a}{\sinh(z_j - v_a) \sinh(z_j + v_a)} f_a \sigma_j^- \phi^a, \quad j = 1, \dots, N \quad (3.11)$$

This equation suggests the Bethe state ϕ (3.8) is an eigenstate of the Gaudin's Hamiltonian H_j , if the set of rapidities $\{v_a\}$ satisfy $f_a = 0$ ($a = 1, \dots, M$), i.e. the Bethe equations:

$$\begin{aligned} & \sum_{k=1}^N [\coth(v_a - z_k) + \coth(v_a + z_k)] \\ = & 2 \tanh v_a - \sum_{k=L,R} [\coth(v_a - c_k) + \coth(v_a + c_k)] + 2 \sum_{\substack{b=1 \\ b \neq a}}^M [\coth(v_a - v_b) + \coth(v_a + v_b)] \end{aligned} \quad (3.12)$$

We have derived the eigenstate and the energy of the XXZ-type Gaudin magnet coupled with impurity spin with different coupling constants c_L and c_R on the boundary.

4 The Knizhnik-Zamolodchikov equation

We consider the KZ equation

$$\nabla_j \Psi = 0, \quad j = 1, \dots, N \quad (4.1)$$

where the differential operator ∇_j is defined by use of Gaudin's Hamiltonians H_j (3.3):

$$\nabla_j = \kappa \frac{\partial}{\partial z_j} - H_j \quad (4.2)$$

and κ is an arbitrary parameter. The integrable condition for a set of the KZ-type differential operators ∇_j ,

$$[\nabla_j, \nabla_k] = 0 \quad \text{for} \quad j, k = 1, \dots, N \quad (4.3)$$

is satisfied by the condition

$$\frac{\partial H_j}{\partial z_k} = \frac{\partial H_k}{\partial z_j} \quad (4.4)$$

Following the idea of [8], we define the hypergeometric function $\mathcal{X}(v|z)$ by a set of differential equations

$$\begin{aligned} \kappa \frac{\partial \mathcal{X}(v|z)}{\partial z_j} &= E_j \mathcal{X}(v|z), \quad j = 1, \dots, N \\ \kappa \frac{\partial \mathcal{X}(v|z)}{\partial v_a} &= f_a \mathcal{X}(v|z), \quad a = 1, \dots, M \end{aligned} \quad (4.5)$$

The integrability of these differential equations follows from the conditions

$$\frac{\partial E_j}{\partial z_k} = \frac{\partial E_k}{\partial z_j}, \quad \frac{\partial E_j}{\partial v_a} = \frac{\partial f_a}{\partial z_j}, \quad \frac{\partial f_a}{\partial v_b} = \frac{\partial f_b}{\partial v_a}. \quad (4.6)$$

In fact, it is straightforward to solve the differential equations (4.5); its solution is the function

$$\begin{aligned}
\mathcal{X}(v|z) &= \prod_a^M (\cosh v_a)^{2/\kappa} [\sinh(v_a - c_L) \sinh(v_a + c_L) \sinh(v_a - c_R) \sinh(v_a + c_R)]^{-1/\kappa} \\
&\quad \times \prod_j^N (\tanh z_j)^{1/2\kappa} [\sinh(z_j - c_L) \sinh(z_j + c_L) \sinh(z_j - c_R) \sinh(z_j + c_R)]^{1/\kappa} \\
&\quad \times \prod_j^N \prod_a^M [\sinh(z_j - v_a) \sinh(z_j + v_a)]^{-1/\kappa} \prod_{a < b}^M [\sinh(v_a - v_b) \sinh(v_a + v_b)]^{2/\kappa}
\end{aligned} \tag{4.7}$$

One can introduce the wavefunction $\Psi(z)$ in a integral form, which has a hypergeometric kernel [31], as

$$\Psi(z) = \oint_C \prod_a^M dv_a \mathcal{X}(v|z) \phi(v|z) \tag{4.8}$$

The integration path C is taken over a closed contour in the Riemann surface such that the integrand resumes its initial value after v_a has described it. The integral function $\Psi(z)$ is in fact a solution of the KZ equation (4.1).

To prove (4.1) we use the fact that the Bethe state ϕ (3.8) satisfies

$$\frac{\partial \phi}{\partial z_j} = \sum_{a=1}^M \left(\frac{\cosh(v_a - z_j)}{\sinh^2(v_a - z_j)} - \frac{\cosh(v_a + z_j)}{\sinh^2(v_a + z_j)} \right) \sigma_j^- \phi_a \tag{4.9}$$

where ϕ_a is defined in (3.9). One sees that the function ϕ_a does not depend on v_a . Then equality (4.1) can be verified as

$$\begin{aligned}
\kappa \frac{\partial \Psi(z)}{\partial z_j} &= \oint_C \left(\kappa \frac{\partial \mathcal{X}}{\partial z_j} \phi + \kappa \mathcal{X} \frac{\partial \phi}{\partial z_j} \right) \prod_a^M dv_a = \oint_C \left(\mathcal{X} E_j \phi + \kappa \mathcal{X} \frac{\partial \phi}{\partial z_j} \right) \prod_a^M dv_a \\
&= H_j \Psi(z) - \oint_C \left(\sum_{a=1}^M \frac{2 \sinh z_j \cosh v_a}{\sinh(z_j - v_a) \sinh(z_j + v_a)} \kappa \frac{\partial \mathcal{X}}{\partial v_a} \sigma_j^- \phi_a - \kappa \mathcal{X} \frac{\partial \phi}{\partial z_j} \right) \prod_a^M dv_a \\
&= H_j \Psi(z) - \kappa \sum_{a=1}^M \oint_C \frac{\partial}{\partial v_a} \left(\mathcal{X} \frac{2 \sinh z_j \cosh v_a}{\sinh(z_j - v_a) \sinh(z_j + v_a)} \right) dv_a \sigma_j^- \phi_a \prod_{b \neq a}^M dv_b \\
&= H_j \Psi(z)
\end{aligned} \tag{4.10}$$

5 Discussion

We have constructed and solved a Gaudin magnet with impurity. The integral representation of the solution for the correspondig KZ equation was obtained and its rational limit is here given by

$$\begin{aligned}
\Psi(z) = & \oint_C dv \prod_a^M [(v_a^2 - c_L^2)(v_a^2 - c_R^2)]^{-1/\kappa} \prod_j^N (z_j)^{1/2\kappa} [(z_j^2 - c_L^2)(z_j^2 - c_R^2)]^{1/\kappa} \\
& \times \prod_j^N \prod_a^M [z_j^2 - v_a^2]^{-1/\kappa} \prod_{a < b}^M [v_a^2 - v_b^2]^{2/\kappa} \\
& \times \prod_a^M \left(\frac{2v_a}{v_a^2 - c_L^2} \sigma_L^- + \frac{2v_a}{v_a^2 - c_R^2} \sigma_R^- + \sum_{j=1}^N \frac{2v_a}{v_a^2 - z_j^2} \sigma_j^- \right) |0\rangle
\end{aligned} \tag{5.1}$$

where $dv = \prod_a^M dv_a$.

Now, if we use our Bethe reference state (2.29) as a particular chiral primary field of the defect conformal field theory [32] based in the $su(2)$ WZW conformal field theory, we conjecture that the integral representation (5.1) should be the candidate for the corresponding M -point correlation function since it is a solution of the KZ equation 4.1. Here we note that the corresponding result without impurity was already obtained by Hikami [?].

In this paper we only considered the $su(2)$ XXZ spin chain case. The generalization to the $su(n)$ should be done from the view point of the Gaudin magnet with impurity [33]. Another also interesting case to be considered is the elliptic XYZ Gaudin magnet with impurity.

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